Collection of Power Flow models: Mathematical formulations

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In its most general form, the optimal power flow (OPF) problem is a cost minimization problem with equality constraints enforcing Kirchhoff's current law, i.e. power balance at each bus in the network, and inequality constraints enforcing physical and stability limits on power generation and flow. Several different formulations exist for the power flow equations, the most popular of which are the polar power-voltage formulation (P), the rectangular power-voltage formulation (R), the rectangular current-voltage (IV), and the DC approximation (DC) which is a linearization of the problem.

Cost functions (1) can be given as quadratic coefficients or as a list of points specifying a piecewise linear function. Piecewise linear functions must be convex at this time.

$$\min \qquad \sum_i \tilde{c}_i(P_i) \tag{1}$$

The following sections provide detail on the mathematical formulations of the models provided in the archive.

1 Nomenclature

The following notation will be used in describing the different OPF formulations. Note that not all parameters and variables will appear in every model, and set $\mathcal{T}$ is typically only used in unit commitment models. Thus, index $t \in \mathcal{T}$ which is used to describe time periods, is not used in all the formulations although it is described and indexed with in these tables.

Note that every line $ijc \in \mathcal{E}$, occurs at most once in this set, that is to say that if $ijc \in \mathcal{E}$, then $jic \notin \mathcal{E}$. The convention used in our models is that a positive flow on a line represents a withdrawal at its source end and an injection at its terminating end. This usage is further clarified in the introductory passage of Section 3.

Set	Description
$\mathcal{N}$	Set of buses in the transmission network
$\mathcal{G}$	Set of generators in the transmission network
$\mathcal{I}$	Set of interfaces in the transmission network
$c \in \mathcal{C}$	Set of transmission connections in network
$t \in \mathcal{T}$	Set of time periods
$\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \times \mathcal{C}$	Set of lines in the transmission network
$\mathcal{E}_i \subseteq \mathcal{E}$	Subset of lines $\mathcal{E}$ belonging to interface $i \in \mathcal{I}$
$\mathcal{G}_i \in \mathcal{G}$	Subset of generators $\mathcal{G}$ at bus $i \in \mathcal{N}$

Table 1: Description of Sets

Parameters	Description
$\tilde{c}_i(\cdot)$	Cost function for generation at bus $i \in \mathcal{N}$
d^P_{it}, d^Q_{it}	Real and reactive power demand at bus i in time $t \in \mathcal{T}$
r_{ijc}, x_{ijc}	Resistance and reactance on line $ijc \in \mathcal{E}$
g^s_i, b^s_i	Shunt conductance and susceptance at bus $i \in \mathcal{N}$
$g^{\mathcal{E}}_{ijc}, b^{\mathcal{E}}_{ijc}$	Conductance and susceptance on line $ijc \in \mathcal{E}$
b^C_{ijc}	Branch charging susceptance of line $ijc \in \mathcal{E}$
τ_{ijc}, ϕ_{ijc}	Transformer tap ratio and angle on line $ijc \in \mathcal{E}$
Y^P_{ij}, Y^Q_{ij}	Real & imaginary component of Y-bus admittance matrix for line $ij \subseteq \mathcal{N} \times \mathcal{N}$
u_{it}	Generator i's on/off status at time $t \in \mathcal{T}$
$\overline{F}^P_{ijc}$	Real power limit on line $ijc \in \mathcal{E}$
$\overline{F}^{\mathcal{I}}_i$	Real power limit on interface $i \in \mathcal{I}$
$\overline{I}_{ijc}$	Current magnitude limit on line $ijc \in \mathcal{E}$
$\underline{P}_i, \overline{P}_i$	Lower and upper active power injection limits of generator $i \in \mathcal{G}$
$\underline{Q}_i, \overline{Q}_i$	Lower and upper reactive power injection limits of generator $i \in \mathcal{G}$
$\underline{U}^{\text{run}}_i, \overline{U}^{\text{run}}_i$	Minimum up and downtime of generator $i \in \mathcal{G}$
$\underline{U}^{\text{ramp}}_i, \overline{U}^{\text{ramp}}_i$	Rampup and rampdown rate of generator $i \in \mathcal{G}$
$\underline{V}_i, \overline{V}_i$	Voltage magnitude lower and upper limits at bus $i \in \mathcal{N}$

Table 2: Description of Parameters

Variables	Description
F^P_{ijct}, F^Q_{ijct}	Real and reactive power flowing on line $ijc \in \mathcal{E}$ in time $t \in \mathcal{T}$
I^P_{ijct}, I^Q_{ijct}	Real and reactive current flowing on line $ijc \in \mathcal{E}$ in time $t \in \mathcal{T}$
P_{it}, Q_{it}	Real and reactive power generated at bus $i \in \mathcal{N}$ in time $t \in \mathcal{T}$
V_{it}, θ_{it}	Voltage magnitude and angle at bus $i \in \mathcal{N}$ in time $t \in \mathcal{T}$
V^P_{it}, V^Q_{it}	Real & imaginary rectangular voltage components at bus $i \in \mathcal{N}$ in time $t \in \mathcal{T}$
$U^{\text{on}}_{it}, U^{\text{off}}_{it}, U^{\text{run}}_{it}$	Start, stop and run status of generator $i \in \mathcal{G}$ at time $t \in \mathcal{T}$

Table 3: Description of Variables

2 DC models

2.1 DC Optimal Power Flow

The DCOPF approximation (`dcopf.gms`) stems from a simplification of the physics of the transmission network. It linearizes the problem by treating resistance and line losses as negligible, assuming that the per unit voltage magnitude at each bus is 1, and uses small angle approximation for sine and cosine. Small angle approximation assumes that the phase angle difference between the buses, $(\theta_i - \theta_j) \forall i \in \mathcal{N}$, is small enough and approximates the cosine and sine functions with $cos(i) = 1$ and $sin(i) = i$ respectively.

With these assumptions, reactive power in the network becomes zero and can be ignored. Since losses are also zero in this model, the power for each ijc line, F_{ijc}, is only computed once (2) as the flow in the opposite direction F_{jic} is the same. The resulting formulation is a linear model that solves for variables P, θ, F^P subject to node balance constraints (3), interface flow limits (4), limits on the angle difference between connected buses (5), and bound constraints on P and F^P (6). Note that generator statuses are provided and this is a single time-period model, therefore the index $t \in \mathcal{T}$ is not used.

$$
\begin{aligned}
\min_{P,F^P,\theta} \quad & \sum_i \tilde{c}_i(P_i) \\
\text{s.t.} \quad & F^P_{ijc} = \frac{-1}{\tau_{ijc}x_{ijc}}(\theta_j - \theta_i + \phi_{ijc}) && \forall ijc \in \mathcal{E} && (2)\\
& \sum_{k\in G_i} P_k - \sum_{(jc):ijc\in\mathcal{E}} F^P_{ijc} + \sum_{(jc):jic\in\mathcal{E}} F^P_{jic} \\
& \qquad - d^P_i - g^s_i = 0 && \forall i \in \mathcal{N} && (3)\\
& \sum_{ijc,jic\in\mathcal{E}_k} F^P_{ijc} \leq \overline{F}^{\mathcal{I}}_k && \forall k \in \mathcal{I} && (4)\\
& \frac{-\pi}{3} \leq \theta_i - \theta_j \leq \frac{\pi}{3} && \forall (ij) : ijc \in \mathcal{E} && (5)\\
& u_i\underline{P}_i \leq P_i \leq u_i\overline{P}_i && \forall i \in \mathcal{G} \\
& -\overline{F}^P_{ijc} \leq F^P_{ijc} \leq \overline{F}^P_{ijc} && \forall ijc \in \mathcal{E} && (6)
\end{aligned}
$$

Additionally, we include the Shift Factor formulation of the DCOPF problem, which models flows on lines with respect to power injections and withdrawals at each bus. This requires a dense $|\mathcal{E}| \times |\mathcal{N}|$ coefficient matrix, which can be generated using the scripts in the data models of the archive.

2.2 DC using Shift Factor Matrices

The DC power flow model can also be used to compute the sensitivities of branch flows to changes in nodal real power injections, sometimes called injection shift factors [2]. These matrices are typically large and dense, $|\mathcal{E}| \times |\mathcal{N}|$, and each element describes the expected change in real power flow on a line, $\mathcal{E}$, as it reacts to a 1 unit increase in the power injection at bus $\mathcal{N}$, under the strict assumption that the additional unit of power is based on some slack distribution. Additional detail about shift matrices can be found in the Matpower manual.

Through use of the shift matrix, we can eliminate bus voltage angles, θ, as an intermediate variable, and the shift matrix model (`dcopf_shift.gms`) is reduced to (7-10), where H is the shift matrix. Due to the matrix density however, this formulation is not recommended over the standard DCOPF formulation, especially in large models.

$$
\begin{aligned}
\min_{P,F^P} \quad & \sum_i \tilde{c}_i(P_i) \\
\text{s.t.} \quad & F^P_{ijc} \quad = \quad \sum_{k \in \mathcal{N}} H_{ijck} \sum_{l \in \mathcal{G}_k} \left(P_l - d^P_k - g^s_i \right) && \forall ijc \in \mathcal{E} && (7) \\
& \sum_{k \in G_i} P_k - d^P_i - g^s_i \quad = \quad 0 && \forall i \in \mathcal{N} && (8) \\
& \sum_{ijc, jic \in \mathcal{E}_k} F^P_{ijc} \quad \leq \quad \overline{F}^{\mathcal{I}}_k && \forall k \in \mathcal{I} && (9) \\
& u_i \underline{P}_i \quad \leq P_i \leq \quad u_i \overline{P}_i && \forall i \in \mathcal{G} && \\
& -\overline{F}^P_{ijc} \quad \leq F^P_{ijc} \leq \quad \overline{F}^P_{ijc} && \forall ijc \in \mathcal{E} && (10)
\end{aligned}
$$

2.3 Unit commitment DC model

`uc_dc.gms` within the model archive is a DC model which incorporates unit commitment, allowing the modeler to choose the set of dispatched generators for each time period, and is typically useful when the OPF model needs to be solved over a continuous block of time. When solved over multiple time periods, $t \in \mathcal{T}$, the model includes additional generator operational constraints such as generator ramping, minimum up-time and minimum down-time. Equations (11 - 14) are equivalent to (2 - 5) with index $t \in \mathcal{T}$, and unit commitment variables $U = \{U^{\text{on}}, U^{\text{off}}, U^{\text{run}}\}$ are factored into generator power limits in (15). Binary variable relationships are modeled in (16), minimum up and down time in (17-18), and generator ramping conditions in (19-20), while (21) defines the remaining variable bounds. The constraints in (16-18) for the generator unit commitment problem is based on the work done by Hedman, O'Neill and Oren in [1].

$$
\begin{aligned}
& \min_{P,F^P,\theta,U} \quad \sum_t \tilde{c}_i(P_{it}) \\
\text{s.t.} \quad & F^P_{ijct} \quad = \quad \frac{-1}{\tau_{ijc} x_{ijc}} (\theta_{jt} - \theta_{it} + \phi_{ijc}) && \forall ijc \in \mathcal{E}, t \in \mathcal{T} && (11) \\
& \sum_{k \in \mathcal{G}_i} P_{kt} - \sum_{(jc):ijc \in \mathcal{E}} F^P_{ijct} \\
& \qquad + \sum_{(jc):jic \in \mathcal{E}} F^P_{jict} - d^P_{it} - g^s_{it} \; = \; 0 && \forall i \in \mathcal{N}, t \in \mathcal{T} && (12) \\
& \sum_{ijc, jic \in \mathcal{E}_k} F^P_{ijct} \quad \leq \quad \overline{F}^{\mathcal{I}}_k && \forall k \in \mathcal{I}, t \in \mathcal{T} && (13) \\
& \frac{-\pi}{3} \quad \leq \theta_{it} - \theta_{jt} \leq \quad \frac{\pi}{3} && \forall (ij) : ijc \in \mathcal{E}, t \in \mathcal{T} && (14) \\
& U^{\text{run}}_{it} * \underline{P}_i \quad \leq P_{it} \leq \quad U^{\text{run}}_{it} * \overline{P}_i && \forall i \in \mathcal{G}, t \in \mathcal{T} && (15)
\end{aligned}
$$

$$U^{\text{on}}_{it} - U^{\text{off}}_{it} \quad = \quad U^{\text{run}}_{it} - U^{\text{run}}_{i,t-1} \qquad \forall i \in \mathcal{G}, t \in \mathcal{T} \quad (16)$$

$$\sum_{t_0 = t - \overline{U}^{\text{run}}_i + 1}^{t} U^{\text{on}}_{i,t_0} \quad \leq \quad U^{\text{run}}_{it} \qquad \forall i \in \mathcal{G}, t \in \mathcal{T} \quad (17)$$

$$U^{\text{run}}_{it} \quad \leq \quad 1 - \sum_{t_0 = t - \underline{U}^{\text{run}}_i + 1}^{t} U^{\text{off}}_{i,t_0} \qquad \forall i \in \mathcal{G}, t \in \mathcal{T} \quad (18)$$

$$P_{it} \quad \leq \quad P_{i,t-1} + U^{\text{run}}_{it}\overline{U}^{\text{ramp}}_i + U^{\text{on}}_{it}(\overline{P}_i - \overline{U}^{\text{ramp}}_i) \qquad \forall i \in \mathcal{G}, t \in \mathcal{T} \quad (19)$$

$$P_{i,t-1} \quad \leq \quad P_{i,t} + U^{\text{run}}_{it}\underline{U}^{\text{ramp}}_i + U^{\text{off}}_{it}(\overline{P}_i - \underline{U}^{\text{ramp}}_i) \qquad \forall i \in \mathcal{G}, t \in \mathcal{T} \quad (20)$$

$$-\overline{F}^P_{ijc} \leq F^P_{ijct} \leq \overline{F}^P_{ijc} \qquad \forall ijc \in \mathcal{E}, t \in \mathcal{T}$$

$$U^{\text{on}}_{it}, U^{\text{off}}_{it}, U^{\text{run}}_{it} \in \{0, 1\} \qquad \forall i \in \mathcal{G}, t \in \mathcal{T} \quad (21)$$

3 AC models

In the true physics of an AC power flow model, power transmitted through lines $ijc \in \mathcal{E}$ may experience loss, and this is reflected in the calculations of active and reactive power. That is to say that the power leaving bus $i \in \mathcal{N}$ on line $ijc \in \mathcal{E}$ may not necessarily equal the power entering the bus $j \in \mathcal{N}$ on the other end, i.e. F_{ijc} may not equal $-F_{jic}$. Recall that the convention used in these models is that a positive flow on a line, F_{ijc} represents a withdrawal at the source $i \in \mathcal{N}$ and an injection at its terminating end $j \in \mathcal{N}$. In this chapter, we provide three formulations of the AC power flow model, namely polar power-voltage in Section 3.1, rectangular power-voltage in 3.2 and rectangular current-voltage in 3.3.

3.1 Polar Power-Voltage Formulation (P)

The polar power-voltage formulation (`polar_acopf.gms`) uses the polar form of complex quantities and explicitly uses sines and cosines in the power flow constraints. Variables Q_i and V_i model the reactive power support provided by generators and the bus voltage at bus $i \in \mathcal{N}$ respectively, while reactive power on a line is modeled by F^Q_{ijc}.

Due to possible line losses as mentioned above, real power flow on lines $ijc \in \mathcal{E}$ which are approximated in the DCOPF by (2) is modeled here by (22 - 23), while (24 - 25) computes the line's reactive power flow. Node balance equations are updated in (26 - 27), limits on line interface flows and bus angle differences are provided in (28 - 29) respectively, and (30) defines bounds on the remaining variables.

$$
\begin{aligned}
&\min_{P,Q,F^P,F^Q,\theta,V} \quad \sum_i \tilde{c}_i(P_i) \\
\text{s.t.} \quad & F^P_{ijc} = \frac{1}{\tau^2_{ijc}} g^{\mathcal{E}}_{ijc} V_i^2 \\
&\quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_i - \theta_j - \phi_{ijc}) + b^{\mathcal{E}}_{ijc} \sin(\theta_i - \theta_j - \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (22) \\
& F^P_{jic} = g^{\mathcal{E}}_{ijc} V_j^2 \\
&\quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_j - \theta_i + \phi_{ijc}) + b^{\mathcal{E}}_{ijc} \sin(\theta_j - \theta_i + \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (23) \\
& F^Q_{ijc} = -\frac{1}{\tau^2_{ijc}} \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2} \right) V_i^2 \\
&\quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_i - \theta_j - \phi_{ijc}) - b^{\mathcal{E}}_{ijc} \sin(\theta_i - \theta_j - \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (24) \\
& F^Q_{jic} = -\left(b_{ijc} + \frac{b^C_{ijc}}{2} \right) V_j^2 \\
&\quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_j - \theta_i + \phi_{ijc}) - b^{\mathcal{E}}_{ijc} \sin(\theta_j - \theta_i + \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (25) \\
& \sum_{k \in \mathcal{G}_i} P_k - \sum_{(jc):ijc \in \mathcal{E}} F^P_{ijc} - \sum_{(jc):jic \in \mathcal{E}} F^P_{ijc} - d^P_i - (V_i)^2 g^s_i \ = \ 0 && \forall i \in \mathcal{N} \quad (26) \\
& \sum_{k \in \mathcal{G}_i} Q_k - \sum_{(jc):ijc \in \mathcal{N}} F^Q_{ijc} - \sum_{(jc):jic \in \mathcal{E}} F^Q_{ijc} - d^Q_i + (V_i)^2 b^s_i \ = \ 0 && \forall i \in \mathcal{N} \quad (27) \\
& \sum_{ijc,jic \in \mathcal{E}_k} F^P_{ijc} \quad \leq \quad \overline{F}^{\mathcal{I}}_k && \forall k \in \mathcal{I} \quad (28) \\
& \frac{-\pi}{3} \quad \leq \theta_i - \theta_j \leq \quad \frac{\pi}{3} && \forall (ij) : ijc \in \mathcal{E} \quad (29) \\
& u_i \underline{P}_i \ \leq P_i \leq \ u_i \overline{P}_i \quad , \quad u_i \underline{Q}_i \ \leq Q_i \leq \ u_i \overline{Q}_i \qquad \forall i \in \mathcal{G} \\
& \underline{V}_i \leq V_i \leq \overline{V}_i \quad \forall i \in \mathcal{E} \quad ; \quad -\overline{F}^P_{ijc} \leq F^P_{ijc} \leq \overline{F}^P_{ijc} \quad \forall ijc \in \mathcal{E} && (30)
\end{aligned}
$$

3.2 Rectangular Power-Voltage Formulation (R)

The second AC formulation we provide (`rect_acopf.gms`) uses the rectangular form of complex quantities, resulting in quadratic power flow constraints with respect to these quantities. Unlike the polar formulation, the sines and cosines are of constant parameters and the bus voltage is separated into real and imaginary parts, that is $V_i = \left((V_i^P)^2 + (V_i^Q)^2 \right) \ \forall i \in \mathcal{N}$. Therefore, equations (22 - 27) from the polar model which define real and reactive power on lines and node balance equations are rewritten in the retangular fomulation as (31-36). Additionally, the voltage magnitude limit is no longer a simple bound contraint but is enforced by the quadratic inequality in (37). Similar to (28-30) in the polar formulation, interface limits are imposed in (38) and variable bounds are defined in (39), while bus angle limits are not explicitly imposed in this formulation.

$$
\begin{aligned}
\min_{P,Q,F^P,F^Q,V} \quad & \sum_i \tilde{c}_i\left(P_i\right) \\
\text{s.t.} \quad & F^P_{ijc} = \frac{1}{\tau^2_{ijc}} g^{\mathcal{E}}_{ijc}\left((V^P_i)^2 + (V^Q_i)^2\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(g_{ijc}\cos(\phi_{ijc}) - b^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_i V^P_j + V^Q_i V^Q_j\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(b^{\mathcal{E}}_{ijc}\cos(\phi_{ijc}) + g^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_j V^Q_i - V^P_i V^Q_j\right) && \forall ijc \in \mathcal{E} \quad (31) \\
& F^P_{jic} = g_{ijc}\left((V^P_j)^2 + (V^Q_j)^2\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(g_{ijc}\cos(\phi_{ijc}) + b^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_j V^P_i + V^Q_j V^Q_i\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(b^{\mathcal{E}}_{ijc}\cos(\phi_{ijc}) - g^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_i V^Q_j - V^P_j V^Q_i\right) && \forall ijc \in \mathcal{E} \quad (32) \\
& F^Q_{ijc} = -\frac{1}{\tau^2_{ijc}}\left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2}\right)\left((V^P_i)^2 + (V^Q_i)^2\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}\cos(\phi_{ijc}) - b^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_j V^Q_i - V^P_i V^Q_j\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(b^{\mathcal{E}}_{ijc}\cos(\phi_{ijc}) + g^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_i V^P_j + V^Q_i V^Q_j\right) && \forall ijc \in \mathcal{E} \quad (33) \\
& F^Q_{jic} = -\left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2}\right)\left((V^P_i)^2 + (V^Q_i)^2\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}\cos(\phi_{ijc}) + b^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_i V^Q_j - V^P_j V^Q_i\right) \\
& \quad - \frac{1}{\tau_{ijc}}\left(b^{\mathcal{E}}_{ijc}\cos(\phi_{ijc}) - g^{\mathcal{E}}_{ijc}\sin(\phi_{ijc})\right)\left(V^P_j V^P_i + V^Q_j V^Q_i\right) && \forall ijc \in \mathcal{E} \quad (34) \\
& \sum_{k\in\mathcal{G}_i} P_k - \sum_{(jc):ijc\in\mathcal{E}} F^P_{ijc} - \sum_{(jc):jic\in\mathcal{E}} F^P_{ijc} - d^P_i \\
& \qquad - \left((V^P_i)^2 + (V^Q_i)^2\right) g^s_i \quad = \quad 0 && \forall i \in \mathcal{N} \quad (35) \\
& \sum_{k\in\mathcal{G}_i} Q_k - \sum_{(jc):ijc\in\mathcal{E}} F^Q_{ijc} - \sum_{(jc):jic\in\mathcal{E}} F^Q_{ijc} - d^Q_i \\
& \qquad + \left((V^P_i)^2 + (V^Q_i)^2\right) b^s_i, \quad = \quad 0 && \forall i \in \mathcal{N} \quad (36) \\
& \underline{V}^2_i \le (V^P_i)^2 + (V^Q_i)^2 \le \overline{V}^2_i && \forall i \in \mathcal{N} \quad (37) \\
& \sum_{ijc,jic\in\mathcal{E}_k} F^P_{ijc} \quad \le \quad \overline{F}^{\mathcal{I}}_k && \forall k \in \mathcal{I} \quad (38)
\end{aligned}
$$

$$u_i\underline{P}_i \leq P_i \leq u_i\overline{P}_i \quad , \quad u_i\underline{Q}_i \leq Q_i \leq u_i\overline{Q}_i \qquad \forall i \in \mathcal{G}$$
$$-\overline{F}^P_{ijc} \leq F^P_{ijc} \leq \overline{F}^P_{ijc} \quad \forall ijc \in \mathcal{E} \tag{39}$$

3.3 Rectangular Current-Voltage Formulation (IV)

The third AC model presented here is the rectangular current-voltage formulation (`iv_acopf.gms`) which considers the flow of current instead of power on a line. Therefore, the model computes real and reactive current on a line, $\{I^P_{ijc}, I^Q_{ijc}\}\ \forall ijc \in \mathcal{E}$, instead of $\{F^P_{ijc}, F^Q_{ijc}\}\ \forall ijc \in \mathcal{E}$ which is the real and reactive power on a line. Similar to the rectangular power-voltage model in Section 3.2, the IV formulation uses the rectangular form of complex quantities, $V_i = \left((V^P_i)^2 + (V^Q_i)^2\right)\ \forall i \in \mathcal{N}$. Therefore, line flow constraints are once again quadratic in nature with constant sine and cosine quantities.

Equations (40-43) define real and reactive current flow on a line, and (44-45) define the node balance constraints. Equations (46-47) impose bounds on the voltage magnitude and current magnitude respectively, and (48) defines other variable bound constraints. Note that the current formulation does not contain interface flow constraints because converting current to power is counter-intuitive in a current-based formulation. In addition, interface constraints were historically used to control LMP prices while current based formulations were used to consider network stability. The IV formualtion also imposes limits on current magnitude in (47) instead of using simple variable bounds on the real such as in the rectangular model ((39) as we find this data is more easily available.

$$\min_{P,Q,I^P,I^Q,V} \quad \sum_i \tilde{c}_i(P_i)$$

$$\text{s.t.} \quad I^P_{ijc} = \frac{1}{\tau^2_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^P_i - \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2}\right)V^Q_i\right)$$
$$- \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^P_j - b^{\mathcal{E}}_{ijc}V^Q_j\right)\cos\left(\phi_{ijc}\right)$$
$$+ \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^Q_j + b^{\mathcal{E}}_{ijc}V^P_j\right)\sin(\phi_{ijc}) \qquad \forall ijc \in \mathcal{E} \tag{40}$$

$$I^P_{jic} = \left(g^{\mathcal{E}}_{ijc}V^P_j - \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2}\right)V^Q_j\right)$$
$$- \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^P_i - b^{\mathcal{E}}_{ijc}V^Q_i\right)\cos(-\phi_{ijc})$$
$$+ \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^Q_i + b^{\mathcal{E}}_{ijc}V^P_i\right)\sin(-\phi_{ijc}) \qquad \forall ijc \in \mathcal{E} \tag{41}$$

$$
\begin{aligned}
I^Q_{ijc} = &\frac{1}{\tau^2_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^Q_i + \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2}\right)V^P_i\right) \\
&- \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^Q_j + b^{\mathcal{E}}_{ijc}V^P_j\right)\cos\left(\phi_{ijc}\right) \\
&- \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^P_j - b^{\mathcal{E}}_{ijc}V^Q_j\right)\sin\left(\phi_{ijc}\right) && \forall ijc \in \mathcal{E} \quad (42)
\end{aligned}
$$

$$
\begin{aligned}
I^Q_{jic} = &\frac{1}{\tau^2_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^Q_i + \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2}\right)V^P_i\right) \\
&- \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^Q_i + b^{\mathcal{E}}_{ijc}V^P_i\right)\cos(-\phi_{ijc}) \\
&- \frac{1}{\tau_{ijc}}\left(g^{\mathcal{E}}_{ijc}V^P_i - b^{\mathcal{E}}_{ijc}V^Q_i\right)\sin(-\phi_{ijc}) && \forall ijc \in \mathcal{E} \quad (43)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k \in \mathcal{G}_i} P_k - d^P_i - V^P_i\left(\sum_{(jc):ijc \in \mathcal{E}} I^P_{ijc} + \sum_{(jc):jic \in \mathcal{E}} I^P_{ijc}\right) \\
&\quad - V^Q_i\left(\sum_{(jc):ijc \in \mathcal{E}} I^Q_{ijc} + \sum_{(jc):jic \in \mathcal{E}} I^Q_{ijc}\right) \\
&\quad - \left((V^P_i)^2 + (V^Q_i)^2\right) g^s_i \;=\; 0 && \forall i \in \mathcal{N} \quad (44)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k \in \mathcal{G}_i} Q_k - d^Q_i + V^P_i\left(\sum_{(jc):ijc \in \mathcal{E}} I^Q_{ijc} + \sum_{(jc):jic \in \mathcal{E}} I^Q_{ijc}\right) \\
&\quad - V^Q_i\left(\sum_{(jc):ijc \in \mathcal{E}} I^P_{ijc} + \sum_{(jc):jic \in \mathcal{E}} I^P_{ijc}\right) \\
&\quad + \left((V^P_i)^2 + (V^Q_i)^2\right) b^s_i \;=\; 0 && \forall i \in \mathcal{N} \quad (45)
\end{aligned}
$$

$$
\underline{V}^2_i \leq (V^P_i)^2 + (V^Q_i)^2 \leq \overline{V}^2_i \qquad \forall i \in \mathcal{N} \qquad (46)
$$

$$
\left(\left(I^P_{ijc}\right)^2 + \left(I^Q_{ijc}\right)^2\right) \leq \overline{I}^2_{ijc} \qquad \forall k \in \mathcal{I} \qquad (47)
$$

$$
u_i\underline{P}_i \leq P_i \leq u_i\overline{P}_i \quad , \quad u_i\underline{Q}_i \leq Q_i \leq u_i\overline{Q}_i \qquad \forall i \in \mathcal{G} \qquad (48)
$$

Note that the current flow equations (40-43) are linear and that (44-46) involve quadratic and linear terms. This would lead us to hope that since the Hessian of the current-voltage constraints is constant, we might see some benefit in solution time, though in practice, this has not been found to be the case in general. Additionally, because this model limits apparent current rather than apparent power on lines, the solutions tend to be slightly different than the other ACOPF models.

3.4 Y-bus formulations

The Y-bus formulations of the above AC models, `ybus_polar_acopf.gms`, `ybus_rect_acopf.gms` and `ybus_iv_acopf.gms` use the Y-bus admittance matrix to calculate in the node balance equations, instead of explicit line flow variables $F_i^P jc, F_i^Q jc$ for example. For discussion on how to compute the Y-bus admittance matrix, please refer to Bergen and Vittal's text on Power Systems Analysis [3].

Although the Y-bus matrix formulation benefits by eliminating line flow parameters in node balance constraints, it loses this benefit in the case where line flow variables still need to be defined in order to maintain line flow limits. For this reason, we find that the standard line power (SLP) models (`polar_acopf.gms`, `rect_acopf.gms`, `iv_acopf.gms`) typically outperform the Y-bus models due to the additional computation required. This document only provides the mathematical formulation for the Y-bus Polar-Power voltage model as the other two formulations are very similar in implementation and can be inferenced from here.

$$
\begin{aligned}
&\min_{P,Q,F^P,F^Q,\theta,V} \quad && \sum_i \tilde{c}_i(P_i) \\
\text{s.t.} \quad & F^P_{ijc} = \frac{1}{\tau^2_{ijc}} g^{\mathcal{E}}_{ijc} V_i^2 \\
& \quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_i - \theta_j - \phi_{ijc}) + b^{\mathcal{E}}_{ijc} \sin(\theta_i - \theta_j - \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (49) \\
& F^P_{jic} = g^{\mathcal{E}}_{ijc} V_j^2 \\
& \quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_j - \theta_i + \phi_{ijc}) + b^{\mathcal{E}}_{ijc} \sin(\theta_j - \theta_i + \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (50) \\
& F^Q_{ijc} = -\frac{1}{\tau^2_{ijc}} \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2} \right) V_i^2 \\
& \quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_i - \theta_j - \phi_{ijc}) - b^{\mathcal{E}}_{ijc} \sin(\theta_i - \theta_j - \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (51) \\
& F^Q_{jic} = -\left(b_{ijc} + \frac{b^C_{ijc}}{2} \right) V_j^2 \\
& \quad - \frac{1}{\tau_{ijc}} V_i V_j \left(g^{\mathcal{E}}_{ijc} \cos(\theta_j - \theta_i + \phi_{ijc}) - b^{\mathcal{E}}_{ijc} \sin(\theta_j - \theta_i + \phi_{ijc}) \right) && \forall ijc \in \mathcal{E} \quad (52)
\end{aligned}
$$

$$\sum_{k\in\mathcal{G}_i} P_k - d_i^P - (V_i)^2 g_i^s$$
$$- V_i \sum_{j\in\mathcal{N}} \Big(V_j Y_{ij}^P \cos(\theta_i - \theta_j) - Y_{ij}^Q \sin(\theta_i - \theta_j) \Big) = 0 \qquad \forall i \in \mathcal{N} \quad (53)$$
$$\sum_{k\in\mathcal{G}_i} Q_k - d_i^Q + (V_i)^2 b_i^s$$
$$- V_i \sum_{j\in\mathcal{N}} \Big(V_j Y_{ij}^P \sin(\theta_i - \theta_j) - Y_{ij}^Q \cos(\theta_i - \theta_j) \Big) = 0 \qquad \forall i \in \mathcal{N} \quad (54)$$
$$\sum_{ijc,jic\in\mathcal{E}_k} F_{ijc}^P \leq \overline{F}_k^{\mathcal{I}} \qquad \forall k \in \mathcal{I} \quad (55)$$
$$\frac{-\pi}{3} \leq \theta_i - \theta_j \leq \frac{\pi}{3} \qquad \forall (ij) : ijc \in \mathcal{E} \quad (56)$$
$$u_i \underline{P}_i \leq P_i \leq u_i \overline{P}_i \quad , \quad u_i \underline{Q}_i \leq Q_i \leq u_i \overline{Q}_i \qquad \forall i \in \mathcal{G}$$
$$\underline{V}_i \leq V_i \leq \overline{V}_i \quad \forall i \in \mathcal{E} \quad ; \quad -\overline{F}_{ijc}^P \leq F_{ijc}^P \leq \overline{F}_{ijc}^P \quad \forall ijc \in \mathcal{E} \quad (57)$$

3.5 Initial conditions of AC OPF models

Due to the difficulty of solving AC models, we have provided multiple starting point options for the SLP models where the variables are initialized using different methods. This is determined by the GAMS model option `ic=#`, and takes inputs 1, 2... 9. We explain some of these options below.

1. `ic=0`: [Midpoint]
   $V_i = \frac{\overline{V}+\underline{V}}{2}, \theta_i = 0 \qquad \forall i \in \mathcal{N}$
   $P_k = \frac{\overline{P}_k+\underline{P}_k}{2}$, $Q_k = \frac{\overline{Q}_k+\underline{Q}_k}{2} \qquad \forall k \in \mathcal{G}$
2. `ic=1`: [Random]
   $V_i = \text{Uniform}(\overline{V}, \underline{V}), \theta_i = \text{Uniform}(-\pi, \pi) \qquad \forall i \in \mathcal{N}$
   $P_k = \text{Uniform}(\overline{P}_k, \underline{P}_k), Q_k = \text{Uniform}(\overline{Q}_k, \underline{Q}_k) \qquad \forall k \in \mathcal{G}$
3. `ic=2`: [Flat]
   $V_i = 1, \theta_i = 0 \qquad \forall i \in \mathcal{N}$
   $P_k = 0, Q_k = 0 \qquad \forall k \in \mathcal{G}$
4. `ic=3`: [Random/AC]
   $V_i = \text{Uniform}(\overline{V}, \underline{V}), \theta_i = \text{Uniform}(-\pi, \pi) \qquad \forall i \in \mathcal{N}$
   P_k, Q_k derived from node balance constraints (model dependent) with V_i, θ_i defined $\forall i \in \mathcal{N}, k \in \mathcal{G}$.
5. `ic=4`: [DC/AC]
   $V_i = 1 \qquad \forall i \in \mathcal{N}$
   θ_i, P_k initialized solving DCOPF (2-6) $\forall i \in \mathcal{N}, k \in \mathcal{G}$.
   Q_k derived from node balance constraints (model dependent) with V_i, θ_i defined $\forall i \in \mathcal{N}, k \in \mathcal{G}$.

6. `ic=5`: [DC-/AC]
   $V_i = 1$, θ_i initialized solving DCOPF (2-6) $\forall i \in \mathcal{N}, k \in \mathcal{G}$.
   P_k, Q_k derived from node balance constraints (model dependent) with V_i, θ_i defined $\forall i \in \mathcal{N}, k \in \mathcal{G}$.

7. `ic=6`: [Decoupled]
   V_i, θ_i, P_k, Q_k initialized by solving the Decoupled AC model (72-77, 78-82).

8. `ic=7`: [DCLoss]
   $V_i = 1 \qquad \forall i \in \mathcal{N}$
   θ_i, P_k initialized solving DCOPF (2-6) with option `--lineloss=1.055` $\forall i \in \mathcal{N}, k \in \mathcal{G}$.
   Q_k derived from node balance constraints (model dependent) with V_i, θ_i defined $\forall i \in \mathcal{N}, k \in \mathcal{G}$.

3.6 Unit commitment AC models

The model archive provides 3 unit commitment AC models, `uc_polar`, `uc_rect` and `uc_iv` which are unit commitment versions of the above SLP models. When solving for $t \in \mathcal{T}$, the generators can be turned on/off throughout such that the set of dispatched generators may change for each time period. Much like the UC_DC model in 2.3, additional constraints include generator ramping, minimum up-time and minimum down-time, and unit commitment variables $U = \{U^{\text{on}}, U^{\text{off}}, U^{\text{run}}\}$ are factored into generator power limits. In the case of the polar model, the unit commitment model adds dimension $t \in \mathcal{T}$ to variables in equations (22 - 29), and adds additional binary handling variables, as shown below.

$$\min_{P,Q,F^P,F^Q,\theta,V} \quad \sum_{it} \tilde{c}_i(P_{it})$$

$$\min_{P,Q,F^P,F^Q,\theta,V} \quad \sum_{i} \tilde{c}_i(P_i)$$

$$\text{s.t.} \quad F^P_{ijct} = \frac{1}{\tau^2_{ijc}} g^{\mathcal{E}}_{ijc} V^2_{it} - \frac{1}{\tau_{ijc}} V_{it} V_{jt} \left(g^{\mathcal{E}}_{ijc} \cos(\theta_{it} - \theta_{jt} - \phi_{ijc}) + b^{\mathcal{E}}_{ijc} \sin(\theta_{it} - \theta_{jt} - \phi_{ijc}) \right) \qquad \forall ijc \in \mathcal{E}, t \in \mathcal{T} \quad (58)$$

$$F^P_{jict} = g^{\mathcal{E}}_{ijc} V^2_{jt} - \frac{1}{\tau_{ijc}} V_{it} V_{jt} \left(g^{\mathcal{E}}_{ijc} \cos(\theta_{jt} - \theta_{it} + \phi_{ijc}) + b^{\mathcal{E}}_{ijc} \sin(\theta_{jt} - \theta_{it} + \phi_{ijc}) \right) \qquad \forall ijc \in \mathcal{E}, t \in \mathcal{T} \quad (59)$$

$$F^Q_{ijct} = -\frac{1}{\tau^2_{ijc}} \left(b^{\mathcal{E}}_{ijc} + \frac{b^C_{ijc}}{2} \right) V^2_{it} - \frac{1}{\tau_{ijc}} V_{it} V_{jt} \left(g^{\mathcal{E}}_{ijc} \cos(\theta_{it} - \theta_{jt} - \phi_{ijc}) - b^{\mathcal{E}}_{ijc} \sin(\theta_{it} - \theta_{jt} - \phi_{ijc}) \right) \qquad \forall ijc \in \mathcal{E}, t \in \mathcal{T} \quad (60)$$

$$F^Q_{jict} = -\left(b_{ijc} + \frac{b^C_{ijc}}{2} \right) V^2_{jt} - \frac{1}{\tau_{ijc}} V_{it} V_{jt} \left(g^{\mathcal{E}}_{ijc} \cos(\theta_{jt} - \theta_{it} + \phi_{ijc}) - b^{\mathcal{E}}_{ijc} \sin(\theta_{jt} - \theta_{it} + \phi_{ijc}) \right) \qquad \forall ijc \in \mathcal{E}, t \in \mathcal{T} \quad (61)$$

$$\sum_{k\in\mathcal{G}_i} P_{kt} - \sum_{(jc):ijc\in\mathcal{E}} F^P_{ijct} - \sum_{(jc):jic\in\mathcal{E}} F^P_{ijct} - d^P_{it} - (V_{it})^2 g^s_i \ = 0 \qquad \forall i\in\mathcal{N}, t\in\mathcal{T} \quad (62)$$

$$\sum_{k\in\mathcal{G}_i} Q_{kt} - \sum_{(jc):ijc\in\mathcal{N}} F^Q_{ijct} - \sum_{(jc):jic\in\mathcal{E}} F^Q_{ijct} - d^Q_{it} + (V_{it})^2 b^s_i \ = 0 \qquad \forall i\in\mathcal{N}, t\in\mathcal{T} \quad (63)$$

$$\sum_{ijc,jic\in\mathcal{E}_k} F^P_{ijct} \quad \le \quad \overline{F}^{\mathcal{I}}_k \qquad \forall k\in\mathcal{I}, t\in\mathcal{T} \quad (64)$$

$$\frac{-\pi}{3} \quad \le \theta_{it} - \theta_{jt} \le \quad \frac{\pi}{3} \qquad \forall (ij): ijc\in\mathcal{E}, \ t\in\mathcal{T} \quad (65)$$

$$U^{\text{run}}_{it} * \underline{Q}_i \quad \le Q_{it} \le \quad U^{\text{run}}_{it} * \overline{Q}_i \qquad \forall i\in\mathcal{G}, t\in\mathcal{T} \quad (66)$$

and(15 − 20)

$$0 \le P_{it} \le \overline{P}_i \quad , \quad \min(0,\underline{Q}_i) \le Q_{it} \le \overline{Q}_i \qquad \forall i\in\mathcal{G}, t\in\mathcal{T}$$

$$\underline{V}_i \le V_{it} \le \overline{V}_i \qquad \forall i\in\mathcal{E}, t\in\mathcal{T}$$

$$-\overline{F}^P_{ijc} \le F^P_{ijct} \le \overline{F}^P_{ijc} \qquad \forall ijc\in\mathcal{E}, t\in\mathcal{T}$$

$$U^{\text{on}}_{it}, U^{\text{off}}_{it}, U^{\text{run}}_{it} \in \{0,1\} \qquad \forall i\in\mathcal{G}, t\in\mathcal{T} \quad (67)$$

4 Decoupled AC Model

Due to the nature of the non-convexity in the power flow equations, large scale ACOPF models are challenging to solve, and research continues to seek algorithms that can provide fast, scalable, and robust solution techniques. Approximations, decompositions and engineering judgment are commonly used to obtain practical solutions. Given that realistic sized models depend on “good” initial conditions to satisfy reasonable computational time and robust convergence, this suggests that if we have a starting point that is fairly close to a feasible ACOPF solution, it may be possible to improve solution time and convergence. The decoupled ACOPF model (`polar_decoupled.gms`) may potentially serve this purpose by providing an initial starting point for the full ACOPF. Note that the model archive also contains `ybus_polar_decoupled.gms` which uses the y-bus admittance matrix as disccused in 3.4.

4.1 The Power Flow Equations

The power flow equations decribe the power system network operating point in steady state and hence are based on complex phasor representation of voltage-current relationships at each bus. Using polar coordinates for complex voltages and rectangular representation for complex power, the active and reactive power flow node balance equations at bus $i\in\mathcal{N}$ are formulated by (68 - 69) respectively.

$$P_i = P_i - d_i^P = \sum_{k=1}^{n} (|V_i||V_k|G_{ik}cos(\theta_i - \theta_k) + |V_i||V_k|B_{ik}sin(\theta_i - \theta_k)) \qquad \forall i \in \mathcal{N} \quad (68)$$

$$Q_i = Q_i - d_i^Q = \sum_{k=1}^{n} \{|V_i||V_k|G_{ik}sin(\theta_i - \theta_k) - |V_i||V_k|B_{ik}cos(\theta_i - \theta_k)\} \qquad \forall i \in \mathcal{N} \quad (69)$$

The most commonly used technique to solve the non-linear power flow equations above is Newton-Raphson. This method requires derivatives of the power flow equations with respect to voltage magnitudes and angles, and is referred to as the power flow Jacobian matrix, The power flow formulation using the Jacobian matrix using (70).

$$\begin{bmatrix} J11 & J12 \\ J21 & J22 \end{bmatrix} \begin{bmatrix} \triangle\theta \\ \triangle V \end{bmatrix} = \begin{bmatrix} \triangle P \\ \triangle Q \end{bmatrix}$$

Where

$$J11 = \frac{\partial P^{bus}}{\partial \theta}, J12 = \frac{\partial P^{bus}}{\partial V}, J21 = \frac{\partial Q^{bus}}{\partial \theta}, J22 = \frac{\partial Q^{bus}}{\partial V} \quad (70)$$

In most real world applications, other assumptions can be made on the range of both parameters and solution variables. Two inequalities that commonly appear in many power flows applications are:

$$\frac{x}{r} \gg 1 \quad \text{and also} \quad |\theta_i - \theta_k| < 20^o$$

Both of these relationships prove to have small values in $J12$ and $J21$ submatrix, and these small values suggest a weak link between active power and voltage magnitude, as well as reactive power and voltage angle. Therefore, if $J12$ and $J21$ in (70) are neglected in the power flow calculation, one obtains the Fast Decoupled Load Flow, for which the linearized relations are:

$$\begin{aligned} J11 \times \triangle\theta &= \triangle P \\ J22 \times \triangle V &= \triangle Q \end{aligned} \quad (71)$$

We employ this idea to decompose the full AC optimal power flow into two subproblems.

4.2 Model

Since voltage magnitudes have weak impact on active power injections, voltage magnitudes may be treated as constant values when solving the active power subproblem. Similarly, voltage angles are assumed to weakly impact reactive power injections. This suggests employing voltage angles as constant values for reactive power subproblem. The constant values (voltage magnitudes) for active power subproblem are needed to be specified to start P-Q decoupling. These starting points are a solution of power flow problem having voltage phasors and apparent power injections of each bus. The number of rows for bus, branch and generator vectors are $|\mathcal{N}|$,$|\mathcal{E}|$ and $|\mathcal{G}|$ respectively.

$P-\theta$ subproblem

For the $P-\theta$ subproblem, the voltage magnitude at each bus is fixed as a constant value, denoted as $\widehat{V}$, while reactive power balance equations are neglected in the constraint set. The solution vector for the standard $P-\theta$ OPF problem consists of an $|\mathcal{N}|\times 1$ vector of voltage angles θ_i , $\forall i \in \mathcal{N}$, and a $|\mathcal{G}|\times 1$ vector of generator active injections P_i, $i \in \mathcal{G}$. The decision variable $x_{P\theta}$ has dimension $\big(|\mathcal{N}|+|\mathcal{G}|\big)\times 1$.

$$x_{P\theta} = \begin{bmatrix}\theta \\ P\end{bmatrix}$$

The objective function is to minimize a summation of each generator quadratic or piecewise-linear cost function (72) subject to (73 - 77).

$$\min_{Q,V} \sum_{i\in\mathcal{N}} \tilde{c}_i(P_i) \tag{72}$$

$$\text{s.t.} \quad g_1(x) = P_i(\theta_i, \widehat{V}_i) + d_i^P + (V_i)^2 g_i^s - P_i = 0, \qquad \forall i \in \mathcal{N} \tag{73}$$

$$h_f(x) = |\ F_{ijc}(\theta_{ij}, \widehat{V}_{ij})\ | - l_{ijc} \leq 0 \qquad \forall ijc \in \mathcal{E} \tag{74}$$

$$h_t(x) = |\ F_{jic}(\theta_{ij}, \widehat{V}_{ij})\ | - l_{jic} \leq 0 \qquad \forall ijc \in \mathcal{E} \tag{75}$$

$$\frac{-\pi}{3} \leq \theta(i) - \theta(j) \leq \frac{\pi}{3} \qquad \forall (ij) : ijc \in \mathcal{E} \tag{76}$$

$$\underline{P}_i \leq P_i \leq \overline{P}_i \qquad \forall i \in \mathcal{G} \tag{77}$$

$Q-V$ subproblem

For the $Q-V$ subproblem, the voltage angles are fixed as a constant value denoted $\widehat{\theta}$. The equality power flow equations for the active power are neglected in the constraint set. The optimization vector x_{QV} for the standard $Q-V$ OPF problem consists of the $|\mathcal{N}|\times 1$ vector of voltage magnitude V_i , $i \in \mathcal{N}$ and the $|\mathcal{G}|\times 1$ vectors of generator reactive injections Q_i , $i \in \mathcal{G}$. The dimension of the decision variable x_{QV} is $\big(|\mathcal{N}|+|\mathcal{G}|\big)\times 1$.

$$x_{QV} = \begin{bmatrix}V \\ Q\end{bmatrix}$$

The objective function is to minimize a transmission line loss which is a function of voltage phasors (78) subject to (79-82).

$$\min \sum_{i\in\mathcal{N}} \left(V_i^2 g_i^s + \sum_{(jc):ijc,jic\in\mathcal{E}} F_{ijc} \right) \tag{78}$$

$$\text{s.t.} \quad g_2(x) = Q_i(\widehat{\theta}_i, V_i) + d_i^Q - (V_i)^2 b_i^s - Q_i = 0 \qquad \forall i \in \mathcal{N} \tag{79}$$

$$h_f(x) = |\ F_{ijc}(\widehat{\theta}_{ij}, V_{ij})\ | - l_{ijc} \leq 0 \qquad \forall ijc \in \mathcal{E} \tag{80}$$

$$h_t(x) = |\ F_{jic}(\widehat{\theta}_{ij}, V_{ij})\ | - l_{jic} \leq 0 \qquad \forall ijc \in \mathcal{E} \tag{81}$$

$$\underline{Q}_i \leq Q_i \leq \overline{Q}_i \quad \forall i \in \mathcal{G} \quad , \quad \underline{V}_i \leq V_i \leq \overline{V}_i \quad \forall i \in \mathcal{N} \tag{82}$$

Note that inequality constraints for line flow limit are included in both problems to represent the security requirements on the system.

Using the model

Decoupled AC OPF can be solved iteratively until an optimal solution converges to serve as an initial condition for the full AC OPF, and each subproblem uses the flat initial condition (x.l=1) for first interation, and optimal solutions of previous subproblems after first iteration. This can be invoked from the command line with the following options:

- --iter=#
  Specifies the number of iterations. The default is 1.

For each formulation, three different models are provided: one version where power flows on lines are handled by explicitly defining them as variables, one condensed version where these equality constraints are removed and power flows are incorporated into the node balance equations, and one version which calculates the node admittance matrix which is used to calculate power or current injections at each bus. We use the convention that a positive flow on a line represents a withdrawal at its source end and an injection at its terminating end.

5 Map of formulation to GAMS notation

This section provides a mapping of the nomenclature used in this document to the actual set, parameter and variables names used in the GAMS code.

Set	**GAMS notation**
$\mathcal{N}$	`i, j, bus`
$\mathcal{G}$	`gen, gen1`
$\mathcal{I}$	`interface`
$c \in \mathcal{C}$	`circuit, c`
$t \in \mathcal{T}$	`t`
$\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \times \mathcal{C}$	`line(i,j,c)`
$\mathcal{E}_i \subseteq \mathcal{E}$	`interfacemap(interface,i,j,c)`
$\mathcal{G}_i \in \mathcal{G}$	`atBus(gen,i)`

Table 4: Mapping of Sets

Parameters	GAMS notation
$\tilde{c}_i(\cdot)$	c_obj (equation)
d_{it}^P, d_{it}^Q	Pd(i,t), Qd(i,t)
r_{ijc}, x_{ijc}	r(i,j,c), x(i,j,c)
g_i^s, b_i^s	gs(i), bs(i)
$g_{ijc}^{\mathcal{E}}, b_{ijc}^{\mathcal{E}}$	g(i,j,c), b(i,j,c)
b_{ijc}^C	bc(i,j,c)
τ_{ijc}, ϕ_{ijc}	ratio(i,j,c), angle(i,j,c)
Y_{ij}^P, Y_{ij}^Q	ybus(i,j,'real'), ybus(i,j,'imag')
u_{it}	status(gen,t)
$\overline{F}_{ijc}^P$	rateA(i,j,c)
$\overline{F}_i^{\mathcal{I}}$	interfaceLimit(interface)
$\overline{I}_{ijc}$	currentrate(i,j,c)
$\underline{P}_i, \overline{P}_i$	Pmin(gen), Pmax(gen)
$\underline{Q}_i, \overline{Q}_i$	Qmin(gen), Qmax(gen)
$\underline{U}_i^{\text{run}}, \overline{U}_i^{\text{run}}$	minuptime(gen), mindowntime(gen)
$\underline{U}_i^{\text{ramp}}, \overline{U}_i^{\text{ramp}}$	rampup(gen), rampdown(gen)
$\underline{V}_i, \overline{V}_i$	minVm(i), maxVm(i)

Table 5: Mapping of Parameters

Variables	Description
F_{ijct}^P, F_{ijct}^Q	V_LineP(i,j,c,t), V_LineQ(i,j,c,t)
I_{ijct}^P, I_{ijct}^Q	V_LineIr(i,j,c,t), V_LineIq(i,j,c,t)
P_{it}, Q_{it}	V_P(gen,t), V_Q(gen,t)
V_{it}, θ_{it}	V_V(i,t), V_Theta(i,t)
V_{it}^P, V_{it}^Q	V_real(i,t), V_imag(i,t)
$U_{it}^{\text{on}}, U_{it}^{\text{off}}, U_{it}^{\text{run}}$	V_startup(gen,t), V_shutdown(gen,t), V_genstatus(gen,t)

Table 6: Mapping of Variables

References

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